# Elucidation of the idea of some strong NP-hardness proofs

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**Abstract**

Here we presents an elucidation of the idea of the strong NP-hardness proofs of some scheduling problems with variable job processing times presented in our papers. In the original papers, we provided the relevant fundamental calculations and additionally also brief descriptions of the ideas of the proofs. We assumed that more detailed descriptions were redundant for the considered strong NP-hardness proofs, since the calculations were given. However, we have been informed that few readers focused only on the descriptions of the ideas omitting the calculations. Therefore, the objective of this note is to clearly point out the relation between the ideas and the calculations.

Although there are no new results in this note, we hope it will help to understand our proofs of the strong NP-hardness (and first of all how the transformations were obtained), if some calculation details are confusing or overlooked during reading the original papers.

**Key words:** Scheduling; Learning effect; Aging effect; Computational complexity

## 1 Introduction

First of all, there are no new results nor changes concerning the original papers in this note, it is only an elucidation of the proofs presented in [1], [2], [3], [4], [5]. Namely, it considers the idea of the strong NP-hardness proofs (concerning distribution of parameters $\lambda_i$) presented at the beginning of part "If" in the mentioned papers. Note that the fundamental for the proofs were the calculations and the idea was additional to help the readers find out how the transformations were obtained. Although, we thought that more detailed descriptions of the idea was redundant (since it was provided additionally), we have been informed that few readers focused only on the idea omitting the calculations. Therefore, in the further part of this note, we will elucidate the relation between the idea and the calculations, to help readers understand the relevant assumptions and the flow of the papers.

## 2 Computational complexity

The considered problems with the variable job processing times (i.e., papers) can be divided into two groups. The first concerns the single processor problems with the maximum lateness minimization (or the makespan minimization with deadlines) [1], [2], [3]. In the second group, there are the flowshop scheduling problem with the makespan minimization [4] and the single processor with the makespan minimization and release dates [5].

Further, we will present only the parts "If" of the considered proofs, since only this part caused problems for some readers. Recall that we have to show that if $\lambda_i = 0$ does not hold for all $i = 1, \dots, m$, then at least one job is late ([1], [2], [3]) or the makespan is greater than the given value $C_{\max}(\pi) > y$ ([4], [5]).

**"If."** The answer for 3-Partition problem (3PP) is *no*. Therefore, there is no partition of the set $X'$ of 3PP such that $\sum_{q \in X'_i} x_q = B$ holds for all $i = 1, \dots, m$, thereby $\sum_{q \in X_i} x_q = B$ does not hold for all $i = 1, \dots, m$. Note that $|X_i| = 3$ for $i = 1, \dots, m$ (follows from the property of the optimal schedule) regardless of the partition of 3PP. Recall that $\sum_{q \in X_i} x_q = B + \lambda_i$ for $i = 1, \dots, m$ and from the assumption $\frac{B}{4} < x_q < \frac{B}{2}$ (for $q = 1, \dots, 3m$) follows that $\frac{3}{4}B < \sum_{q \in X_i} x_q < \frac{3}{2}B$, therefore, $\lambda_i \in (-\frac{B}{4}, \frac{B}{2})$. Since the answer for 3PP is *no*, thus, for any partition of the set $\{1, \dots, 3m\}$ into disjoint subsets $X_1, \dots, X_m$, there must exist at least two subsets $X_u$ and $X_w$ ($u \neq w$) such that $\sum_{q \in X_u} x_q \neq \sum_{q \in X_w} x_q$ for $u, w \in \{1, \dots, m\}$ and $u < w$.

### 2.1 The maximum lateness minimization

In this part, we consider the strong NP-hardness proofs of the maximum lateness minimization problems with variable processing times [1] or the makepan minimization problem with variable job processing times and deadlines ([2], [3]), which leads to the same conclusions.

At first, let us present the idea of the proofs, where it is sufficient (from the perspective of the criterion value and taking into consideration the given transformation) to consider only two types of cases, since any distribution of $\lambda_i$ (following the partition of jobs) can be reduced to them (without increasing the criterion value). The parameters of jobs for the transformation from 3PP to the given problems were obtained using similar approach, based on the following cases (here, we have simplified them):

(a) $\lambda_u > 0$, $\lambda_w < 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \dots, w-1$,

(b) $\lambda_u < 0$, $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \dots, w-1$,

where $u, w \in \{1, \dots, m\}$ and $u < w$ and $\lambda_i = 0$ for $i \in \{1, \dots, u-1\} \cup \{w+1, \dots, m\}$, thereby $\sum_{l=u}^{w} \lambda_l = 0$. Furthermore, there was shown that cases (a) and (b) can be analysed separately for searching the transformation and during the proving process.

Regardless of (a) and (b), the relevant calculations in these papers were done for

(a') $\lambda_u > 0$,

(b') $\lambda_u < 0$ and $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \geq 0$),

where for both $\lambda_i = 0$ for $i = 1, \ldots, u-1$.

However, the detailed descriptions of the cases seemed to be redundant in the presence of the calculations and they were omitted in the original paper. Nevertheless, in this short note, we will provide them with the main calculations (for their full derivation, the reader is refereed to the original paper).

#### 2.1.1 The proof in [1]

In this part, we will elucidate the proof presented in [1].

At first analyse cases (here denoted by (a')), where $\lambda_u > 0$ (in fact $\lambda_u \geq 1$ since they are integer numbers) and $\lambda_i = 0$ for $i = 1, \ldots, u-1$. Recall, the completion time of the last job in $E_{u+1}$ is

$$C_{E_{u+1}} \quad > \quad d_{E_{u+1}} + (N+3)u\lambda_u - \frac{3}{4}Bu.$$

Since $\lambda_u > 0$ and $N = mB > \frac{3}{4}B$, then $C_{E_{u+1}} > d_{E_{u+1}}$, thereby $L_{\max} > 0$.

Now let us recall the second part of the proof (here denoted by (b')). In these general opposite cases (complement to (a')) $\lambda_u < 0$ and $\lambda_i = 0$ for $i = 1, \ldots, u-1$ ($w \leq m$). It can be easily observed that there must exist the set $X_w$ ($u < w$) for which $\sum_{l=u}^{w} \lambda_l \geq 0$, i.e., the first (starting from $u$) for which $\sum_{l=u}^{w} \lambda_l < 0$ does not hold. In other words, we have $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ (since $\lambda_u < 0$ and $\lambda_w > 0$).

Note that the completion time of the last job in $E_i$ for $i = u+1, \ldots, w+1$ (and $w \leq m-1$) can be estimated:

$$C_{E_i} \quad > \quad d_{E_i} + (N+3)\sum_{l=u}^{i-1} l\lambda_l - \frac{3}{4}B(i-1).$$

On this basis and taking into consideration $\sum_{i=u}^{w} i\lambda_i = w\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l$, the completion time of the last job in $E_{w+1}$ (where $w \leq m-1$) is:

$$C_{E_{w+1}} \quad > \quad d_{E_{w+1}} + (N+3)\Big(w\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l\Big) - \frac{3}{4}Bw.$$

Note that $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ and $\sum_{i=u}^{w} \lambda_i \geq 0$. However, the above is minimal if $\sum_{i=u}^{w} \lambda_i = 0$, thus, we have

$$C_{E_{w+1}} \quad > \quad d_{E_{w+1}} - (N+3)\sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l - \frac{3}{4}Bw > d_{E_{w+1}} + (N+3) - \frac{3}{4}Bw > d_{E_{w+1}},$$

for $w \leq m-1$, thereby $L_{\max} > 0$. If $w = m$, then the completion time of the last scheduled job in $X_m$ can be estimated as follows:

$$C_{X_m} \quad > \quad D + (N+3)\Big(w\sum_{i=u}^{m}\lambda_i - \sum_{i=u}^{m-1}\sum_{l=u}^{i}\lambda_l\Big) - \frac{3}{4}Bm.$$

Again the above is minimal if $\sum_{i=u}^{m} \lambda_i = 0$ (since $w = m$) and $C_{X_m} > D + (N+3) - \frac{3}{4}Bm > D$, thereby $L_{\max} > 0$.

Therefore, for any feasible distribution of $\lambda_i$ (where are least two $\lambda_i$ are not zero), we have $L_{\max} > 0$.

On this basis, it can be easily observed for cases (a') that $L_{\max} > 0$ regardless of the the values of $\lambda_i$ for $i > u$, thus, without loss of generality we can assume that $\sum_{l=u}^{i} \lambda_i > 0$ for $i = u, \ldots, w-1$, $\lambda_i = 0$ for $i = w+1, \ldots, m$ and $\sum_{l=u}^{w} \lambda_l = 0$. Thereby, the analysed cases (a') can be reduced to (a). On the other hand, for cases (b'), the job completion time $C_{E_{w+1}}$ (or $C_{X_m}$ if $w = m$) is minimal if $\sum_{l=u}^{w} \lambda_l \geq 0$ is minimal, thereby for $\sum_{l=u}^{w} \lambda_l = 0$. Moreover, the values of $\lambda_i$ for $i > w$ are immaterial, hence we can assume $\lambda_i = 0$ for $i > w$. Thereby, without loss of generality the analysed cases (b') can be reduced to (b).

### 2.1.2 The proof in [2]

In this part, we will elucidate the proof presented in [2] similarly as for [1].

Obviously for case (a'), we have $C_{e_{u+1}} > \bar{d}_{e_{u+1}}$ and without loss of generality it can be reduced to (a).

However, for case (b'), we have showed the calculations assuming that $\lambda_i = 0$ for $i > w$. It follows from the fact that values $\lambda_i$ for $i > w$ are important only for terms, where we have $\sum_{i=u}^{w} l\lambda_i = w\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l$, and they are minimized if $\sum_{i=u}^{w} \lambda_i = 0$, i.e., in this proof for $C_{e_{w+1}}$. Therefore, it was easy to observe that $C_{e_{w+1}} > \bar{d}_{e_{w+1}}$ regardless of $\lambda_i$ for $i > w$ based on the calculations concerning $C_{\max}$; they were similar (see Eq. (11) and Eq. (12) in [2]). Thus, the relevant calculations were omitted. However, in this note we will provide them for $w < m$ (based in Eq. (11) for $k = w+1$):

$$C_{e_{w+1}} \quad > \quad \bar{d}_{e_{w+1}} + \beta\Big[p_e\Big(w\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l\Big) - \frac{1}{4}mB(3Hm + H)\Big].$$

Note that $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ and $C_{e_{w+1}}$ is minimized if $\sum_{l=u}^{w} \lambda_l \geq 0$ is minimized, thereby $\sum_{l=u}^{w} \lambda_l = 0$, hence we obtain $C_{e_{w+1}} > \bar{d}_{e_{w+1}}$ (similarly as for $C_{\max}$)

regardless of $\lambda_i$ for $i > w$. Thus, the distribution of $\lambda_i$ for $i > w$ is immaterial. Obviously, if $w = m$, then $i > w$ is also immaterial and $C_{\max}$ is analysed, which is greater than $y$ as shown in [2]. Therefore, without loss of generality for this proof, cases (b') can be reduced to (b).

### 2.1.3 The proof in [3]

In [3] similarly as in [2], we have omitted calculations concerning $C_{e_{w+1}}$ and assumed for (b') (without loss of generality) that $\lambda_i = 0$ for $i > w$; it follows from the same reasons (see also Eq. (26) and Eq. (27) in [3]). However to show it clearly we will provide the calculations for $w < m$ (based on Eq. (26) for $k = w + 1$ and recall that $\sum_{l=u}^{w} \lambda_l \geq 0$):

$$\begin{aligned} C_{e_{w+1}} &> \bar{d}_{e_{w+1}} + \beta 2\Big(p_e + 3H + B\Big)\left[\sum_{i=u}^{w} i\lambda_i + \sum_{i=u}^{w}\sum_{l=u}^{i-1} \lambda_l\right] - \beta\Big(2mB^2 + m^2B^2\Big) \\ &+ \beta^2 p_e\Big(p_e + 3H + B\Big)\left[\sum_{i=u}^{w} i^2\lambda_i + \sum_{i=u}^{w} 2(i-1)\sum_{l=u}^{i-1}\lambda_l\right] - \beta^2 4m^2Bp_e\Big(B + 4mH + mB^2H\Big). \end{aligned}$$

Recall that

$$\sum_{i=u}^{w} i\lambda_i + \sum_{i=u}^{w}\sum_{l=u}^{i-1} \lambda_l = w\sum_{i=u}^{w} i\lambda_i \geq 0,$$

$$\sum_{i=u}^{w} i^2\lambda_i + \sum_{i=u}^{w} 2(i-1)\sum_{l=1}^{i-1}\lambda_l = w^2\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w}\sum_{l=u}^{i-1}\lambda_i \geq -\sum_{i=u}^{w}\sum_{l=u}^{i-1}\lambda_i,$$

thus, $C_{e_{w+1}} > \bar{d}_{e_{w+1}} + \beta 2m^2B^2\Big(\frac{p_e^2}{4m^2B^2N} - 1\Big) = \bar{d}_{e_{w+1}}$ regardless of $\lambda_i$ for $i > w$. Similarly as for [2], the distribution of $\lambda_i$ for $i > w$ is immaterial. Obviously, if $w = m$, then $i > w$ is also immaterial and $C_{\max}$ is analysed, which is greater than $y$ as shown in [3]. Therefore, without loss of generality for this proof, cases (a') and (b') can be reduced to (a) and (b).

## 2.2 The makespan minimization

In this part, we consider the strong NP-hardness proofs of the makespan minimization problems with variable processing times in the following environments: flowshop [4] and the single processor with release dates [5].

At first, let us present the idea of the proof, where it is sufficient (from the perspective of the criterion value and taking into consideration the given transformation) to consider only two types of cases, since any distribution of $\lambda_i$ (following the partition of jobs) can be reduced to them (without increasing the criterion value). The parameters of jobs for the transformation from 3PP to the given problems were obtained using similar approach, based on the following cases (here, we have simplified them):

(a) $\lambda_u < 0$, $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$,

(b) $\lambda_u > 0$, $\lambda_w < 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$,

where $u, w \in \{1, \ldots, m\}$ and $u < w$ and $\lambda_i = 0$ for $i \in \{1, \ldots, u-1\} \cup \{w+1, \ldots, m\}$, thereby $\sum_{l=u}^{w} \lambda_l = 0$. Furthermore, there was shown that cases (a) and (b) can be analysed separately for searching the transformation and during the proving process. The cases are the same as for [1], [2], [3], but denoted in the reversed order.

First of all, in the cases (a) and (b) in the related papers, it is assumed that $\lambda_i = 0$ for $i \in \{1, \ldots, u-1\} \cup \{w+1, \ldots, m\}$ and $\sum_{l=u}^{w} \lambda_l = 0$, since it does not affect the relation between $C_{\max}(\pi)$ and $y$ (it results from the calculations given in the papers as we will show further). Regardless of (a) and (b), the relevant calculations in these papers were done (in fact) for

(a') $\lambda_w > 0$,

(b') $\lambda_u > 0$ and $\lambda_w < 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \geq 0$),

where for both $\lambda_i = 0$ for $i = w+1, \ldots, m$.

However, the detailed descriptions of the cases seemed to be redundant in the presence of the calculations and they were omitted in the original paper. Nevertheless, in this short note, we will provide them with the main calculations (for their full derivation, the reader is refereed to the original paper). It will be elucidated based on [4] (it is similar for [5]).

At first recall the first part of the proof (here denoted by (a')), where $\lambda_w > 0$ (in fact $\lambda_u \geq 1$ since they are integer numbers) and $\lambda_i = 0$ for $i = w+1, \ldots, m$. Recall the completion time of the last job in $X_w$ on $M_2$ can start not earlier than at $C_{X_w}^{(1)}$ (i.e., $S_{e_{w+1}}^{(1)}$) regardless of the distribution of $\lambda_i$ for $i < w$) and it can be estimated as follows:

$$C_{X_w}^{(2)} \geq S_{e_{w+1}}^{(1)} + V^{(2)}(X_w) = S_{e_{w+1}}^{(1)} + p_{e_{w+1}}^{(1)} + (m-w+1)\lambda_w \geq C_{e_{w+1}}^{(1)} + (m-w+1)\lambda_w > C_{e_{w+1}}^{(1)}.$$

Since $\lambda_w > 0$ and $\lambda_i = 0$ for $i = w+1, \ldots, m$, then $C_{X_i}^{(2)} \geq C_{e_{i+1}}^{(1)} + (m-w+1)\lambda_w > C_{e_{i+1}}^{(1)}$, and we have

$$C_{\max}(\pi) = \max\{C_{e_{m+1}}^{(1)}, C_{X_m}^{(2)}\} + p_{e_{m+1}}^{(2)} \geq C_{e_{m+1}}^{(1)} + (m-w+1)\lambda_w + p_{e_{m+1}}^{(2)} = y + (m-w+1)\lambda_w > y.$$

Note that [4] showed additionally the calculations concerning the minimum values of $C_{X_u}^{(2)}$ (however it was only informative).

Now let us recall the second part of the proof (here denoted by (b')). In these general opposite cases (complement to (a')) $\lambda_w < 0$ and $\lambda_i = 0$ for $i = w+1, \ldots, m$ ($w \leq m$). It can be easily observed that there must exist the set $X_u$ ($u < w$) for which $\sum_{l=u}^{w} \lambda_l \geq 0$, i.e., the first (starting from $w$) for which $\sum_{l=u}^{w} \lambda_l < 0$ does not hold. In other words, we have $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$ (since $\lambda_w < 0$ and $\lambda_u > 0$).

Note that $X_u$ can start on $M_2$ not earlier than at $C^{(1)}_{X_u}$ (i.e., $S^{(1)}_{e_{u+1}}$) (according to the transformation) regardless of the distribution of $\lambda_i$ for $i = 1, \dots, u-1$, i.e., it is minimal for $C^{(1)}_{X_u}$. Therefore, the completion time of the last job in $X_u$ on $M_2$ equals

$$C^{(2)}_{X_u} = S^{(1)}_{e_{u+1}} + V^{(2)}(X_u) = S^{(1)}_{e_{u+1}} + p^{(1)}_{e_{u+1}} + (m-u+1)\lambda_u = C^{(1)}_{e_{u+1}} + (m-u+1)\lambda_u > C^{(1)}_{e_{u+1}}.$$

Following this way and knowing that $\sum_{l=u}^{i}(m-l+1)\lambda_l = (m-i+1)\sum_{l=u}^{i}\lambda_l + \sum_{l=u}^{i-1}\sum_{k=u}^{l}\lambda_k$, we have (for $i = u, \dots, w-1$)

$$C^{(2)}_{X_i} = C^{(1)}_{e_{i+1}} + \sum_{l=u}^{i}(m-l+1)\lambda_l = C^{(1)}_{e_{i+1}} + (m-i+1)\sum_{l=u}^{i}\lambda_l + \sum_{l=u}^{i-1}\sum_{k=u}^{l}\lambda_k.$$

Note that $\sum_{k=u}^{i}\lambda_k > 0$ for $i = u, \dots, w-1$ and $\sum_{k=u}^{w}\lambda_k \geq 0$. However, the above is minimal (for cases (b')) if $\sum_{k=u}^{w}\lambda_k = 0$, hence $C^{(2)}_{X_i} > C^{(1)}_{e_{i+1}}$ (for $i = u, \dots, w-1$). Therefore, we have

$$C^{(2)}_{X_w} = C^{(1)}_{e_{w+1}} + (m-w+1)\sum_{l=u}^{w}\lambda_l + \sum_{l=u}^{w-1}\sum_{k=u}^{l}\lambda_k \geq C^{(1)}_{e_{w+1}} + \sum_{l=u}^{w-1}\sum_{k=u}^{l}\lambda_k > C^{(1)}_{e_{w+1}},$$

$$C^{(2)}_{e_{w+1}} \geq S^{(1)}_{e_{w+2}} + \sum_{l=u}^{w-1}\sum_{k=u}^{l}\lambda_k > S^{(1)}_{e_{w+2}}.$$

Since $\lambda_i = 0$ for $i = w+1, \dots, m$, then $C^{(2)}_{X_i} \geq C^{(1)}_{e_{i+1}} + \sum_{l=u}^{w-1}\sum_{k=u}^{l}\lambda_k$. Thus, we obtain

$$C_{\max}(\pi) = \max\{C^{(1)}_{e_{m+1}}, C^{(2)}_{X_m}\} + p^{(2)}_{e_{m+1}} \geq C^{(1)}_{e_{m+1}} + \sum_{l=u}^{w-1}\sum_{k=u}^{l}\lambda_k + p^{(2)}_{e_{m+1}} = y + \sum_{l=u}^{w-1}\sum_{k=u}^{l}\lambda_k > y.$$

Observe that $C_{\max}(\pi) > y$ regardless of the distribution of $\lambda_i$ for $i < u$ and the criterion value is minimized if $\sum_{k=u}^{w}\lambda_k \geq 0$ is minimized, thereby for $\sum_{k=u}^{w}\lambda_k = 0$.

Based on the calculations, it can be easily observed for cases (a') that $C_{\max}(\pi) > y$ regardless of the the values of $\lambda_i$ for $i < w$. However, $X_w$ starts later for $\lambda_{w-1} > 0$ than for $\lambda_{w-1} \leq 0$ (and can start later if the same relation is hold for $i < w-1$). However, to be formal, there must exist $X_u$ ($u < w$) such that $\sum_{l=u}^{w}\lambda_l = 0$.Thus, without loss of generality for this proof, we can assume that $\lambda_l = 0$ for $l = 1, \dots, u-1$ and $\sum_{l=u}^{i}\lambda_l < 0$ for $i = u, \dots, w-1$ (which have the minimal contribution to $C^{(2)}_{X_w}$) and $\sum_{l=u}^{w}\lambda_l = 0$. Thus, the analysed cases (a') can be reduced to (a).

Additionally for cases (b') note that $C_{\max} > y$ regardless of the distribution of $\lambda_i$ for $i < u$ and the criterion value is minimized if $\sum_{l=u}^{w}\lambda_l \geq 0$ is minimized, thereby for $\sum_{l=u}^{w}\lambda_l = 0$. Observe also that $X_u$ starts later for $\lambda_{u-1} > 0$ than for $\lambda_{u-1} \leq 0$ (and can start later if the same relation is hold for $i < u-1$). Therefore, $\lambda_i = 0$ for $i = 1, \dots, u-1$ have the smallest contribution to $C^{(2)}_{X_u}$ (actually none contribution). Thus, without loss of generality, for this proof we can assume $\sum_{l=u}^{w} \lambda_l = 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$ and $\lambda_i = 0$ for $i \in \{1, \ldots, u-1\} \cup \{w+1, \ldots, m\}$. Thereby, without loss of generality the analysed cases (b') can be reduced to (b).

# 3 Conclusions

After the transformation was obtained the calculations were done for (a') and (b') (however, as we have shown the calculations for (a) and (b) were sufficient).

It is worth mentioning that the presented approach supports (based on (a) and (b)) the construction of the transformations such that it is easier to include the terms $\sum_{l=u}^{i} l\lambda_l$ ([1], [2], [3]) or $\sum_{l=u}^{i}(m-l+1)\lambda_l$ ([4], [5]) into the completion times of relevant jobs. It was very helpful during the process of searching the transformation that guarantees $L_{\max} \leq 0$ or $C_{\max} \leq y$ if and only if $\sum_{q \in X_i'} x_q = B$ for $i = 1, \ldots, m$. Furthermore, in the discussed proofs if $L_{\max} > 0$ or $C_{\max}(\pi) > y$ for cases described by (a) and (b), then they still hold for other distributions of $\lambda_i$ (i.e., (a') and (b')).

We hope that above elucidation will help to understand the presented proofs of the strong NP-hardness (and first of all how the transformations were obtained), if some calculation details are confusing or overlooked during reading the original papers.

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