A critical note on "Strong NP-hardness of scheduling problems with learning or aging effect"

Radosław Rudek

Wrocław University of Economics,
Komandorska 118/120, 53-345 Wrocław, Poland
Tel.: +48-71-368-0378; Fax.:+48-71-368-0376
E-mails: radoslaw.rudek@ue.wroc.pl

Abstract

This critical note concerns the paper [A. Janiak, M. Y. Kovalyov, M. Lichtenstein, Strong NP-hardness of scheduling problems with learning or aging effect. Annals of Operations Research, 206:577–583, 2013]. We show that the so called "corrections" presented by Janiak et al. are exaggerated, since they concerns only some informal simplifications in our proofs. The "formalization" of these proofs is given here, which require only few comments that are the same for all the proofs. Moreover, we also prove that the paper by Janiak et al. contains basic algebraic mistakes that were the source of their misleading conclusions resulting in finding artificial errors in our papers. Finally, we highlight some parts of our papers to dispel doubts and to help confused readers to better understand our proofs of the strong NP-hardness.

Key words: Scheduling; Flowshop; Single machine; Learning effect; Aging effect; Computational complexity

1 Introduction

This critical note concerns the paper [6] by Janiak et al., that were trying to show that the strong NP-hardness proofs in our papers ([1], [2],[3]) contained errors, which required their corrections ([6]). Moreover, they claimed that our proofs in [4], [5] had such errors that they were unable to corrected, thereby the computational complexity of the related problems is still an open issue.

Nevertheless, in this paper, we will show that the so called "corrections" presented in [6] are exaggerated, since they concerns only some informal simplifications in our proofs. The "formalization" of these proofs will be shown here, which requires only few comments that are the same for all the proofs. It also show the strength of our approach that supports the strong NP-hardness proving process. Moreover, we will also prove that the claims of Janiak et al. that computational complexity of the problems analysed in [4] and [5] "remains unknown because of another mistake", which they are unable to correct, follows from their own mistake with adding parameters.

Since some simplifications in our proofs were exaggerated by Janiak et al. [6] to the so called "mistake", thus, we feel that readers deserve more detailed elucidation, which will be given.

2 Problems formulation

Following Janiak et al. [6], we will analyse the problems in the given order:

1. $FP2|LE_{step}|C_{\max}$ [1],
2. $1|r_j, LE_{step}|C_{\max}$ [2],
3. $1|p_j(v) = a_j v|L_{\max}$ [3],
4. $1|p_j(v) = a - b_j v|L_{\max}$ [3],
5. $1|\bar{d}_j, \tilde{p}_j(v) = p_j(1 + \beta \sum_{l=1}^{v-1} p_{[l]})^{\alpha_j}, \alpha_j \in \{0, 1\}|C_{\max}$ [4],
6. $1|\bar{d}_j, AE|C_{\max}$ [5].

The definitions of the problems are provided in the related papers and also in [6]. All of them are based on a reduction from the 3-Partition problem (3PP), which is defined as follows.

3-PARTITION (3PP): Given $3m+1$ positive integers $x_1, \ldots, x_{3m}$ and B such that $\sum_{j=1}^{3m} x_j = mB$ and $\frac{B}{4} < x_q < \frac{B}{2}$ for $j = 1, \ldots, 3m$, is there a partition of the set $X = \{1, \ldots, m\}$ into m disjoint subsets X_i such that $\sum_{j \in X_i} x_j = B$ for $i = 1, \ldots, m$?

3 "Mistake" and its "corrections"

The so called "mistake" found by Janiak et al. [6] concerns the part "if" in our proofs, where the answer "no" for 3PP implied "no" for the decision problem, which computational complexity was proved. Namely, there is no partition of the set X of 3PP such that $\sum_{j \in X_i} x_j = B$ holds for all $i = 1, \ldots, m$. In other words, if $\sum_{j \in X_i} x_j = B + \lambda_i$ (where $\lambda_i \in (-\frac{B}{4}, \frac{B}{2})$) for $i = 1, \ldots, m$, then there are at least two values such that $\lambda_u \neq 0$ and $\lambda_w \neq 0$, where $1 \leq u < w \leq m$.

First of all, Janiak et al. provided counter-examples concerning λ_i values, which in their meaning showed errors in our proofs. However, even for their cases all our proofs work, i.e., the answer in the part "if" is "no" if it is "no" for 3PP.

Later on, Janiak et al. provided their own approach to "correct", but in fact they provide only a different notation for our proofs, which is based on our reductions and properties of the optimal solutions. It is worth highlighting that for proving the strong NP-hardness often the most difficult is to find a relevant reduction (in this case they use ours).

In fact so called "mistake" concerns the fact that in our proof we reduced the range of parameters λ_i to the most significant values, which have the relevant impact on the job completion times, i.e., if the answer is "no" for them, then it is also "no" even for a broader range. Now, we are aware that we should give additional comments, but we will provide them here.

Let us recall the cases (a) and (b) in part "if" for [1] and [2]:

(a) $\lambda_u < 0$, $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$,

(b) $\lambda_u > 0$, $\lambda_w < 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$,

where $u, w \in \{1, \ldots, m\}$ and $u < w$ and $\lambda_i = 0$ for $i \in \{1, \ldots, u-1\} \cup \{w+1, \ldots, m\}$, thereby $\sum_{l=u}^{w} \lambda_l = 0$. Furthermore, there was shown that cases (a) and (b) can be analysed separately for searching the transformation and during the proving process.

Note that for [3], [4] and [5], cases (a) and (b) are interchanged:

(a) $\lambda_u > 0$, $\lambda_w < 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$,

(b) $\lambda_u < 0$, $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$,

where $u, w \in \{1, \ldots, m\}$ and $u < w$ and $\lambda_i = 0$ for $i \in \{1, \ldots, u-1\} \cup \{w+1, \ldots, m\}$, $\sum_{l=u}^{w} \lambda_l = 0$.

The so called "mistake" concerns that we provided cases (a) and (b), in other words, we assumed that $\lambda_i = 0$ for $i \in \{1, \ldots, u-1\} \cup \{w+1, \ldots, m\}$, thereby $\sum_{l=u}^{w} \lambda_l = 0$. However, it was a simplification only, and an informal description in the proofs. Now, we are aware that we should provide the general distribution of λ_i and after that we should explained, why we presented the reduced range of this parameters, whereas calculations were relevant for the general cases.

Namely, to better understood the proofs presented in our papers [1], [2], cases (a) and (b) can be provided at the end of the proof, whereas the following more general cases should be given at the beginning:

(a') $\lambda_w > 0$,

(b') $\lambda_u > 0$ and $\lambda_w < 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \geq 0$),

where for both $\lambda_i = 0$ for $i = w+1, \ldots, m$ and $1 \leq u < w \leq m$.

Observe that (a') covers cases, where $\lambda_w > 0$ is the first (starting from m towards 1) non-zero element; the values of λ_i for $i < w$ are arbitrary. On the other hand, (b') covers other cases.

Namely, $\lambda_w < 0$ is the first (starting from m towards 1) non-zero element and λ_u is the first element (starting from m towards 1) such that $\sum_{l=u}^{w} \lambda_l \geq 0$. Thus, $\lambda_u > 0$ and $\sum_{l=u}^{i} \lambda_i > 0$ for $i = u, \ldots, w-1$; the values of λ_i for $i < u$ are arbitrary. Therefore, (a') and (b') cover all possible distributions of λ_i for $i = 1, \ldots, m$. Furthermore, cases (a') and (b') can be analysed separately, i.e., there is no sense to analyse them simultaneously (see [1]).

It is the same for [3], [4] and [5], where the general cases that cover all possible distributions of λ_i are given as follows:

(a") $\lambda_u > 0$,

(b") $\lambda_u < 0$ and $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \geq 0$),

where $1 \leq u < w \leq m$ and for both $\lambda_i = 0$ for $i = 1, \ldots, u-1$.

In other words, (a") covers cases, where $\lambda_u > 0$ is the first (starting from 1 towards m) non-zero element; the values of λ_i for $i > u$ are arbitrary. On the other hand, (b") covers other cases. Namely, $\lambda_u < 0$ is the first (starting from 1 towards m) non-zero element and λ_w is the first element (starting from 1 towards m) such that $\sum_{l=u}^{w} \lambda_l \geq 0$. Thus, $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_i < 0$ for $i = u, \ldots, w-1$; the values of λ_i for $i > w$ are arbitrary. It can be observed that (a") and (b") cover all possible distributions of λ_i for $i = 1, \ldots, m$. Some symmetry can be observed to cases (a') and (b').

Observe that the "formalization" of the proofs requires only to provide cases (a') and (b') (for [1] and [2]) or (a") and (b") (for [3], [4] and [5]) instead of (a) and (b). Thus, no more "corrections" are required as it was claimed by Janiak et al. [6]. It is elucidated in details in the next part. Moreover, we will also show that without loose of generality the general cases may be reduced to (a) and (b).

3.1 Problem $FP2|LE_{step}|C_{\max}$ [1]

Janiak et al. [6] provided some new calculations as so called "required corrections" for our proof in [1], whereas the following is sufficient to see the proof.

Namely, instead of (a) and (b) in [1], it was better to provide (at the beginning of the proof) the following two cases: (a'): $\lambda_w > 0$; (b'): $\lambda_u > 0$ and $\lambda_w < 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \geq 0$); where for both $\lambda_i = 0$ for $i = w+1, \ldots, m$. Since the relevant calculations in [1], where provided taking into consideration general cases (i.e., (a') and (b')), then no changes in calculations are done. Thus, it makes the proof formal and general and no more "corrections" are required. However, to easier follow the proof and the discussion, we recall some of its parts.

Consider case (a') note that regardless of the distribution of λ_i for $i < w$, the first job in X_w can start on M_2 not earlier than $\underline{C_{X_w}^{(1)}} = S_{e_{w+1}}^{(1)}$, thus,

$$C_{X_w}^{(2)} \ \underline{\ \geq\ }\ S_{e_{w+1}}^{(1)} + V^{(2)}(X_w) \geq C_{e_{w+1}}^{(1)} + (m-w+1)\lambda_w > C_{e_{w+1}}^{(1)}.$$

Since $\lambda_w > 0$ and $\lambda_i = 0$ for $i = w+1, \ldots, m$, then $C_{X_i}^{(2)} > C_{e_{i+1}}^{(1)}$, and we have

$$C_{\max} = \max\{C_{e_{m+1}}^{(1)}, C_{X_m}^{(2)}\} + p_{e_{m+1}}^{(2)} \geq C_{e_{m+1}}^{(1)} + (m-w+1)\lambda_w + p_{e_{m+1}}^{(2)} = y + (m-w+1)\lambda_w > y.$$

Consider (b') X_u can start on M_2 not earlier than at $C_{X_u}^{(1)}$ (i.e., $S_{e_{u+1}}^{(1)}$) regardless of the distribution of λ_i for $i < u$, thus,

$$C_{X_u}^{(2)} \ \underline{\ \geq\ }\ S_{e_{u+1}}^{(1)} + V^{(2)}(X_u) = C_{e_{u+1}}^{(1)} + (m-u+1)\lambda_u > C_{e_{u+1}}^{(1)}.$$

Following this way and knowing that $\sum_{l=u}^{i}(m-l+1)\lambda_l = (m-i+1)\sum_{l=u}^{i} \lambda_l + \sum_{l=u}^{i-1}\sum_{k=u}^{l} \lambda_k$, we have (for $i = u, \ldots, w-1$)

$$C_{X_i}^{(2)} \ \underline{\ \geq\ }\ C_{e_{i+1}}^{(1)} + \sum_{l=u}^{i}(m-l+1)\lambda_l = C_{e_{i+1}}^{(1)} + (m-i+1)\sum_{l=u}^{i} \lambda_l + \sum_{l=u}^{i-1}\sum_{k=u}^{l} \lambda_k.$$

Since $\sum_{k=u}^{i} \lambda_k > 0$ for $i = u, \ldots, w-1$, hence $C_{X_i}^{(2)} > C_{e_{i+1}}^{(1)}$. We have

$$\begin{aligned} C_{X_w}^{(2)} &= C_{e_{w+1}}^{(1)} + (m-w+1)\sum_{l=u}^{w} \lambda_l + \sum_{l=u}^{w-1}\sum_{k=u}^{l} \lambda_k \geq C_{e_{w+1}}^{(1)} + \sum_{l=u}^{w-1}\sum_{k=u}^{l} \lambda_k > C_{e_{w+1}}^{(1)}, \\ C_{e_{w+1}}^{(2)} &\geq S_{e_{w+2}}^{(1)} + \sum_{l=u}^{w-1}\sum_{k=u}^{l} \lambda_k > S_{e_{w+2}}^{(1)}. \end{aligned}$$

Since $\lambda_i = 0$ for $i = w+1, \ldots, m$, then $C_{X_i}^{(2)} \geq C_{e_{i+1}}^{(1)} + \sum_{l=u}^{w-1}\sum_{k=u}^{l} \lambda_k$. Thus, we obtain

$$C_{\max} = \max\{C_{e_{m+1}}^{(1)}, C_{X_m}^{(2)}\} + p_{e_{m+1}}^{(2)} \geq C_{e_{m+1}}^{(1)} + \sum_{l=u}^{w-1}\sum_{k=u}^{l} \lambda_k + p_{e_{m+1}}^{(2)} = y + \sum_{l=u}^{w-1}\sum_{k=u}^{l} \lambda_k > y.$$

Note that there are no new calculations, since the above is taken from [1] to show that the relations hold. The only changes (underlined) were some additional comments and the replacement "=" with "≥" (typos). Observe also that $C_{\max} > y$ regardless of the distribution of λ_i for $i < u$ and the criterion value is minimized if $\sum_{k=u}^{w} \lambda_k \geq 0$ is minimized, thereby for $\sum_{k=u}^{w} \lambda_k = 0$.

3.2 Problem $1|r_j, LE_{step}|C_{\max}$ [2]

For the proof [2] the following is sufficient to see the proof.

Namely, instead of (a) and (b) in [2], it was better to provide (at the beginning of the proof) the following two cases: (a'): $\lambda_w > 0$; (b'): $\lambda_u > 0$ and $\lambda_w < 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \geq 0$); where for both $\lambda_i = 0$ for $i = w+1, \ldots, m$. Since the relevant calculations in [2], where provided taking into consideration general cases (i.e., (a') and (b')), then no changes in calculations are done. Thus, it makes the proof formal and general and no more "corrections" are required. However, to easier follow the proof, we recall some of its parts.

For case (a'), regardless of the distribution of λ_i for $i < w$, we have

$$C_{X_w}(\pi) \geq r_{e_{w-1}} + 1 + 6(m-w+1)B + \frac{m-w+1}{2m(m+1)}(B + \lambda_w) = r_{e_w} + \frac{m-w+1}{2m(m+1)}\lambda_w > r_{e_w}.$$

Since $\lambda_i = 0$ for $i > w$, then each job e_i $(i > w)$ starts after its release date. Finally, the completion time of X_w is estimated

$$C_{X_m}(\pi) \geq 3(m+1)mB + \frac{1}{4}B + m - 1 + \frac{m-w+1}{2m(m+1)}\lambda_w = r_{e_m} + \frac{m-w+1}{2m(m+1)}\lambda_w > r_{e_m},$$

and it results that $C_{\max}(\pi) = C_{e_m} = \max\{C_{X_m}(\pi), r_{e_m}\} + 1 > r_{e_m} + 1 = y$.

Consider case (b'), where regardless of the distribution of λ_i for $i < w$, the first job in X_u starts not earlier than its release date. Thus, we have

$$C_{X_i}(\pi) \underline{\;\geq\;} r_{e_i} + \frac{1}{2m(m+1)}\left[(m-i+1)\sum_{l=u}^{i} \lambda_l + \sum_{l=u}^{i-1}\sum_{k=u}^{l} \lambda_k\right].$$

Since $\sum_{l=u}^{i} \lambda_l > 0$ for $i = 1, \ldots, w-1$, then $C_{X_i} > r_{e_i}$. Thus, the completion time of X_w is

$$C_{X_w}(\pi) \geq r_{e_w} + \frac{1}{2m(m+1)}\sum_{l=u}^{w-1}\sum_{k=u}^{l} \lambda_k > r_{e_w},$$

and it results that $C_{\max} > r_{e_m} + 1 = y$, since $\lambda_i = 0$ and $C_{X_i} > r_{e_i}$ for $i = w+1, \ldots, m$. Thus, for all the cases the criterion value $C_{\max}(\pi)$ is greater than y.

Note that there are no new calculations, since the above is taken from [2] to show that the relations hold. The only changes (underlined) were some additional comments and the replacement "=" with "≥" (typos).

3.3 Problem $1|p_j(v) = a_j v|L_{\max}$ [3]

Janiak et al. [6] provided a page of so called "required corrections" for our proof, whereas the following is sufficient to see the proof.

Namely, instead of (a) and (b) in [3], it was better to provide (at the beginning of the proof) the following two cases: (a"): $\lambda_u > 0$; (b") $\lambda_u < 0$ and $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \geq 0$); where for both $\lambda_i = 0$ for $i = 1, \ldots, u-1$. Since the relevant calculations in [3], where provided taking into consideration general cases (i.e., (a") and (b")), then no changes in calculations are done. Thus, it makes the proof formal and general and no more "corrections" are required. However, to easier follow the proof, we recall some of its parts.

Consider case (a"). Recall, the completion time of the last job in E_{u+1} is

$$C_{E_{u+1}} \quad > \quad d_{E_{u+1}} + (N+3)u\lambda_u - \frac{3}{4}Bu.$$

Since $\lambda_u > 0$ and $N = mB > \frac{3}{4}B$, then $C_{E_{u+1}} > d_{E_{u+1}}$, thereby $L_{\max} > 0$, <u>regardless of</u> $\underline{\lambda_i}$ <u>for</u> $\underline{i > u}$.

Consider case (b"). The completion time of the last job in E_i for $i = u+1, \ldots, w+1$ (and $w \leq m-1$) can be estimated:

$$C_{E_i} \quad > \quad d_{E_i} + (N+3)\sum_{l=u}^{i-1} l\lambda_l - \frac{3}{4}B(i-1).$$

On this basis and taking into consideration $\sum_{i=u}^{w} i\lambda_i = w\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l$, the completion time of the last job in E_{w+1} (where $w \leq m-1$) is:

$$C_{E_{w+1}} \quad > \quad d_{E_{w+1}} + (N+3)\Big(w\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l\Big) - \frac{3}{4}Bw.$$

Since $\sum_{l=u}^{i} \lambda_l < 0$ for $u \leq i < w$ <u>and the above is minimal for</u> $\underline{\sum_{i=u}^{w} \lambda_i = 0}$, thereby

$$C_{E_{w+1}} \quad > \quad d_{E_{w+1}} - (N+3)\sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l - \frac{3}{4}Bw > d_{E_{w+1}} + (N+3) - \frac{3}{4}Bw > d_{E_{w+1}},$$

for $w \leq m-1$, thereby $L_{\max} > 0$. If $w = m$, then the completion time of the last scheduled job in X_m can be estimated as follows:

$$C_{X_m} \quad > \quad D + (N+3)\Big(w\sum_{i=u}^{m}\lambda_i - \sum_{i=u}^{m-1}\sum_{l=u}^{i}\lambda_l\Big) - \frac{3}{4}Bm > D + (N+3) - \frac{3}{4}Bm > D.$$

Again the above is minimal if $\sum_{i=u}^{m} \lambda_i = 0$ (since $w = m$) and $C_{X_m} > D + (N+3) - \frac{3}{4}Bm > D$, thereby $L_{\max} > 0$.

Note that there are no new calculations, since the above is taken from [3] to show that the relations hold. Only some additional comments are given to support the reader (underlined).

3.4 Problem $1|p_j(v) = a - b_j v|L_{\max}$ [3]

Note that Janiak et al. [6] claimed for the considered problem that "*a minor modification is needed to apply [their] correction for the problem* $1|p_j(v) = a - b_j v|L_{\max}$". However, the relevant part of our proof was omitted in [3], since it was exactly the same as for the problem $1|p_j(v) = av|L_{\max}$, thus, no additional comments are required here.

3.5 Problem $1|\bar{d}_j, \tilde{p}_j(v) = p_j(1 + \beta \sum_{l=1}^{v-1} p_{[l]})^{\alpha_j}, \alpha_j \in \{0,1\}|C_{\max}$ [4]

Janiak et al. [6] claimed that they had found a mistake in our strong NP-hardness proof in [4]. To prove it, they showed that in the part "if" of the proof, the maximum completion time $C_{\max}(T)$ calculated for the following sequence $T = (e_1, \ldots, e_m, 1, \ldots, 3m)$ can be in some cases smaller than the deadline D, i.e., $C_{\max}(T) < D$. We will show, that their claim is incorrect. Namely, they added some numbers incorrectly, and first and foremost, our proof is correct, which will be shown further.

Recall that Janiak et al. calculated the maximum completion time for T as follows:

$$C_{\max}(T) = p_e m + \underline{mB} + \beta P_2(m,B) = D \ \underline{\ - 6m^2 B\ } \ + \beta(P_2(m,B) - P_1(m,B)),$$

where $P_1(B,m)$ and $P_2(m,B)$ are non negative polynomials dependent on m and B. However, they incorrectly summarized values, since $p_j = H + x_j = 2mB + x_j$, but the above is calculated for $p_j = x_j$, which is incorrect (the mistake is underlined).

The correct maximum completion time for the sequence T is as follows:

$$C_{\max}(T) = p_e m + \underline{6m^2 B} + mB + \beta P_2(m,B) = D + \beta(P_2(m,B) - P_1(m,B)),$$

where the missing term in [6] is underlined. The above always hold the relation $C_{\max}(T) > D$ as it is required in our proof in [4].

Janiak et al. to show the incorrectness of our proof, assumed that $P_2(m,B) - P_1(m,B) < 0$, but it is incorrect assumption, which is not even calculated. For the given T (or the part "if"), we have always $P_2(m,B) - P_1(m,B) > 0$. On the other hand, if $P_2(m,B) - P_1(m,B) > 0$, then always $C_{\max}(T) > D$.

In the further part, we will focus on our proof in [4]. Namely, instead of (a) and (b) in [4], it was better to provide (at the beginning of the proof) the following two cases: (a"): $\lambda_u > 0$; (b") $\lambda_u < 0$ and $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \geq 0$); where for both $\lambda_i = 0$ for $i = 1, \ldots, u-1$. Thus, it makes the proof formal and general. However, to easier follow the proof, we recall some of its parts.

Obviously for case (a"), we have $C_{e_{u+1}} > \bar{d}_{e_{u+1}}$ regardless of λ_i for $i > u$ (see [4]).

Consider case (b"). It is easy to observe that $C_{e_{w+1}} > \bar{d}_{e_{w+1}}$ regardless of λ_i for $i > w$. However, the calculations were omitted, since they are similar to $C_{\max}$ (see Eq. (11) and Eq. (12) in [4]). Nevertheless, in this note we will provide them for $w < m$ (based on Eq. (11) for $k = w+1$):

$$C_{e_{w+1}} \ > \ \bar{d}_{e_{w+1}} + \beta\Big[p_e\Big(w\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w-1}\sum_{l=u}^{i}\lambda_l\Big) - \frac{1}{4}mB(3Hm + H)\Big].$$

Note that $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ and $C_{e_{w+1}}$ is minimized if $\sum_{l=u}^{w} \lambda_l \geq 0$ is minimized, thereby $\sum_{l=u}^{w} \lambda_l = 0$, hence we obtain $C_{e_{w+1}} > \bar{d}_{e_{w+1}}$ (similarly as for $C_{\max}$) regardless of λ_i for $i > w$. Thus, the distribution of λ_i for $i > w$ is immaterial. Obviously, if $w = m$, then $i > w$ is also immaterial and $C_{\max}$ is analysed, which is greater than y as shown in [4].

3.6 Problem $1|\bar{d}_j, AE|C_{\max}$ [5]

Janiak et al. [6] claimed that they had found a mistake in our strong NP-hardness proof in [5]. To prove it, they showed that in the part "if" of the proof, the maximum completion time $C_{\max}(T)$ calculated for the following sequence $T = (e_1, \ldots, e_m, 1, \ldots, 3m)$ can be in some cases smaller than the deadline D, i.e., $C_{\max}(T) < D$. We will show, that their claim is incorrect. Namely, they added some numbers incorrectly, and first and foremost, our proof is correct, which will be shown further.

Recall that Janiak et al. calculated the maximum completion time for T as follows:

$$\begin{aligned} C_{\max}(T) &= p_e m + mB + \beta R_1(m,B) + \beta^2 R_2(m,B) \\ &= D \ \underline{\ - 24m^3 B^2\ } \ + \beta(R_1(m,B) - Q_1(m,B)) + \beta^2(R_2(m,B) - Q_2(m,B)), \end{aligned}$$

where $R_1(B,m)$, $R_2(B,m)$, $Q_1(B,m)$ and $Q_2(m,B)$ are non negative polynomials dependent on m and B. However, they incorrectly summarized values, since $p_j = H + x_j = 8m^2B^2 + x_j$, but the above is calculated for $p_j = x_j$, which is incorrect (the result of the mistake is underlined).

The correct maximum completion time for the sequence T is as follows:

$$\begin{aligned} C_{\max}(T) &= p_e m + \underline{24m^3B^2} + mB + \beta R_1(m,B) + \beta^2 R_2(m,B) \\ &= D + \beta(R_1(m,B) - Q_1(m,B)) + \beta^2(R_2(m,B) - Q_2(m,B)), \end{aligned}$$

where the missing term in [6] is underlined. The above always hold the relation $C_{\max}(T) > D$ as it is required in our proof in [5].

Janiak et al. to show the incorrectness of our proof, assumed that $R_1(m,B) - Q_1(m,B)) < 0$ and $\beta^2(R_2(m,B) - Q_2(m,B) < 0$, but it is incorrect assumption, which was not even calculated. For the given T (or the part "if"), we have always $\beta(R_1(m,B) - Q_1(m,B)) + \beta^2(R_2(m,B) - Q_2(m,B)) > 0$ for $4m^2B^2/p_e^2 \le \beta \le 1$ (see [5]), thereby $C_{\max}(T) > D$.

In the further part, we will focus on our proof in [5]. Namely, instead of (a) and (b) in [5], it was better to provide (at the beginning of the proof) the following two cases: (a"): $\lambda_u > 0$; (b") $\lambda_u < 0$ and $\lambda_w > 0$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ (and $\sum_{l=u}^{w} \lambda_l \ge 0$); where for both $\lambda_i = 0$ for $i = 1, \ldots, u-1$. Thus, it makes the proof formal and general. However, to easier follow the proof, we recall some of its parts.

Consider case (a"). Obviously, we have $C_{e_{u+1}} > \bar{d}_{e_{u+1}}$ regardless of λ_i for $i > u$ (as it was shown in [5]).

In [5] similarly as in [4], we have omitted calculations concerning $C_{e_{w+1}}$ and assumed for (b") (without loss of generality) that $\lambda_i = 0$ for $i > w$; it follows from the same reasons (see also Eq. (26) and Eq. (27) in [5]). However to show it clearly we will provide the calculations for $w < m$ (based on Eq. (26) for $k = w+1$ and recall that $\sum_{l=u}^{w} \lambda_l \ge 0$):

$$\begin{aligned} C_{e_{w+1}} &> \bar{d}_{e_{w+1}} + \beta 2\Big(p_e + 3H + B\Big)\left[\sum_{i=u}^{w} i\lambda_i + \sum_{i=u}^{w}\sum_{l=u}^{i-1} \lambda_l\right] - \beta\Big(2mB^2 + m^2B^2\Big) \\ &+ \beta^2 p_e\Big(p_e + 3H + B\Big)\left[\sum_{i=u}^{w} i^2\lambda_i + \sum_{i=u}^{w} 2(i-1)\sum_{l=u}^{i-1}\lambda_l\right] - \beta^2 4m^2Bp_e\Big(B + 4mH + mB^2H\Big). \end{aligned}$$

Observe that

$$\sum_{i=u}^{w} i\lambda_i + \sum_{i=u}^{w}\sum_{l=u}^{i-1} \lambda_l = w\sum_{i=u}^{w} i\lambda_i \ge 0,$$

$$\sum_{i=u}^{w} i^2\lambda_i + \sum_{i=u}^{w} 2(i-1)\sum_{l=1}^{i-1}\lambda_l = w^2\sum_{i=u}^{w}\lambda_i - \sum_{i=u}^{w}\sum_{l=u}^{i-1}\lambda_i \ge -\sum_{i=u}^{w}\sum_{l=u}^{i-1}\lambda_i,$$

thus, $C_{e_{w+1}} > \bar{d}_{e_{w+1}} + \beta 2m^2B^2\Big(\frac{p_e^2}{4m^2B^2N} - 1\Big) = \bar{d}_{e_{w+1}}$ regardless of λ_i for $i > w$. Similarly as for [4], the distribution of λ_i for $i > w$ is immaterial. Obviously, if $w = m$, then $i > w$ is also immaterial and $C_{\max}$ is analysed, which is greater than y as shown in [5]. Therefore, without loss of generality for this proof, cases (a") and (b") can be reduced to (a) and (b).

3.7 An elucidation of a simplification

Based on the calculations for [1] and [2], it can be easily observed for case (a') that $C_{\max} > y$ holds regardless of the the values of λ_i for $i < w$. However, X_w starts later for $\lambda_{w-1} > 0$ than for $\lambda_{w-1} \le 0$ (and can start later if the same relation is hold for $i < w-1$). However, to be formal, there must exist X_u ($u < w$) such that $\sum_{l=u}^{w}\lambda_l \ge 0$. Since λ_i for $i < w$ does not affect the relation $C_{\max} > y$, thus, we may assume that $\sum_{l=u}^{w}\lambda_l = 0$ (it changes nothing in the relation $C_{\max} > y$). Therefore, without loss of generality for these proofs, we may assume that $\lambda_l = 0$ for $l = 1, \ldots, u-1$ and $\sum_{l=u}^{i} \lambda_l < 0$ for $i = u, \ldots, w-1$ and $\sum_{l=u}^{w} \lambda_l = 0$. Thus, the analysed case (a') may be reduced to case, where $\sum_{l=u}^{w} \lambda_l = 0$ (denoted by (a) in [1] and [2]). Additionally for case (b') note that $C_{\max} > y$ holds regardless of the distribution of λ_i for $i < u$ and the criterion value is minimized if $\sum_{l=u}^{w} \lambda_l \geq 0$ is minimized, thereby for $\sum_{l=u}^{w} \lambda_l = 0$. Observe also that X_u starts later for $\lambda_{u-1} > 0$ than for $\lambda_{u-1} \leq 0$ (and can start later if the same relation is hold for $i < u-1$). Therefore, $\lambda_i = 0$ for $i = 1, \ldots, u-1$ have the smallest contribution to $C_{X_u}^{(2)}$ for [1] (and C_{X_u} for [2]). Thus, without loss of generality, for these proofs we may assume $\sum_{l=u}^{w} \lambda_l = 0$ and $\sum_{l=u}^{i} \lambda_l > 0$ for $i = u, \ldots, w-1$ and $\lambda_i = 0$ for $i \in \{1, \ldots, u-1\} \cup \{w+1, \ldots, m\}$. Thereby, without loss of generality for these proof the analysed case (b') may be reduced to a case, where $\sum_{l=u}^{w} \lambda_l = 0$ (denoted by (b) in [1] and [2]). In other words, for the analysed proof, if $C_{\max} > y$ hold for the cases denoted by (a) and (b) (where it is assumed that $\sum_{l=u}^{w} \lambda_l = 0$), it also holds for general (a') and (b').

Similar relations hold for [3], [4] and [5]. It is easy to observe that for case (a") applied for instance to [3], we have $L_{\max} > y$ regardless of λ_i for $i > u$. Since λ_i for $i > u$ are immaterial, then we may assume that $\sum_{l=u}^{w} \lambda_l = 0$, it changes nothing in the given relation. On the other hand, for case (b"), we have $L_{\max} > y$ regardless of λ_i for $i > w$. Furthermore, $\sum_{l=u}^{w} \lambda_l = 0$ has the minimal contribution to the criterion value for λ_i, where $i \in \{u, \ldots, w\}$. Thereby, if $L_{\max}$ holds for (a) and (b) (i.e., $\sum_{l=u}^{w} \lambda_l = 0$) it can be easily observed that it also holds for (a") and (b"), thus, they may be reduced to (a) and (b).

Finally, it can be observed that if we assume $\sum_{l=u}^{w} \lambda_l = 0$ for (a') and (b') and also for (a") and (b"), then they are reduced to cases (a) and (b). Thus, the same cases can be used for different criteria, which is comfortable. Although in the mentioned papers, the simplified cases were considered, these proofs were correct. However, it may seem to be less formal and confusing for some readers. Note that such simplification is sufficient for the discussed problems, but it does not have to hold for other problems.

4 Conclusions

Using so called "corrections" and the approach by Janiak et al. [6], they were unable to apply them for [4], [5], since in fact, they concerned informal simplification in our proofs (reduced ranges of parameters). Therefore, it is sufficient only to "formalize" the proofs (or to elucidate the applied simplification), which required only to provide few comments, i.e., cases (a') and (b') (for [1] and [2]) or (a") and (b") (for [3], [4] and [5]) instead of (a) and (b). Thus, no more "corrections" are required as it was claimed by Janiak et al. [6]. Furthermore, we also showed the general cases can be reduced (without loos of generality) to (a) and (b) for the considered proofs.

References

[1] R. Rudek. Computational complexity and solution algorithms for flowshop scheduling problems with the learning effect. *Computers & Industrial Engineering*, 61:20–31, 2011.

[2] R. Rudek. On single processor scheduling problems with learning dependent on the number of processed jobs. *Applied Mathematical Modelling*, 37:1523–1536, 2013.

[3] R. Rudek. Scheduling problems with position dependent job processing times: computational complexity results. *Annals of Operations Research*, 196:491–516, 2012.

[4] R. Rudek. Some single-machine scheduling problems with the extended sum-of-processing-time based aging effect. *International Journal of Advanced Manufacturing Technology*, 59:299–309, 2012.

[5] R. Rudek. The strong NP-hardness of the maximum lateness minimization scheduling problem with the processing-time based aging effect. *Applied Mathematics and Computation*, 218:6498–6510, 2012.

[6] A. Janiak, M. Y. Kovalyov, M. Lichtenstein. Strong NP-hardness of scheduling problems with learning or aging effect. *Annals of Operations Research*, 206:577–583, 2013.